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Proof of the symmetry of the off-diagonal Hadamard/ Seeley-deWitt's coefficients in C^∞ Lorentzian manifolds by a "local Wick rotation".

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Abstract: Completing the results achieved in a previous paper, we prove the symmetry of Hadamard/Seeley-deWitt off-diagonal coefficients in smooth D-dimensional *Lorentzian* manifolds. This result is relevant because it plays a central rôle in Physics, in particular in the theory of the stress-energy tensor renormalization procedure in quantum field theory in curved spacetime. To this end, it is shown that, in any Lorentzian manifold, a sort of "local Wick rotation" of the metric can be performed provided the metric is a (locally) analytic function of the coordinates and the coordinate are appropriate. No time-like Killing field is necessary. Such a local Wick rotation analytically continues the Lorentzian metric in a neighborhood of any point (more generally, in a neighborhood of a space-like (Cauchy) hypersurface) into a Riemannian metric. The continuation locally preserves geodesically convex neighborhoods. In order to make rigorous the procedure, the concept of a complex pseudo-Riemannian (not Hermitian or Kählerian) manifold is introduced and some features are analyzed. Using these tools, the symmetry of Hadamard/Seeley-deWitt off-diagonal coefficients is proven in Lorentzian analytical manifolds by analytical continuation of the (symmetric) Riemannian heat-kernel coefficients. This continuation is performed in geodesically convex neighborhoods in common with both the metrics. Then, the symmetry is generalized to C^∞ non analytic Lorentzian manifolds by approximating Lorentzian C^∞ metrics by analytic metrics in common geodesically convex neighborhoods.

1 Introduction, generalities and summary of previous results.

1.1. In a previous paper [Mo99c] we have considered the problem of the symmetry of heat-kernel/Seeley-deWitt coefficients, taken off-diagonal, for a second order differential operator A_0 defined in a manifold $\mathcal{M}$.
As is well-known [Wa78, Wa94] that symmetry property assures the validity of some physically very important requirements (e.g. the conservation along the motion) of the

quantum stress-energy tensor in quantum field theory in curved spacetime, whenever such a tensor is renormalized by means of the "point-splitting" procedure. In [Mo99c], we considered the Euclidean case whereas, within this paper we want to deal with the Lorentzian case which is much more interesting on physical grounds.
From now on, $\mathcal{M}$ denotes a (real, Hausdorff, paracompact, connected, orientable) D-dimensional C^∞ manifold endowed with a non-singular either Lorentzian (namely, the signature is $(-,+,\cdots,+)$) or Riemannian metric, $\mathbf{g}$ [1]. (In the next section we shall consider also complex manifolds.) The operator A_0 has the form

$$A_0 = -\Delta + V \;\;:\;\; C_0^\infty(\mathcal{M}) \to L^2(\mathcal{M}, d\mu_g)\,, \tag{1}$$

whenever the metric is Riemannian. Conversely, in the Lorentzian case, the operator A_0 has the form

$$A_0 = -\Delta + V \;\;:\;\; D(\mathcal{M}) \to C^\infty(\mathcal{M})\,, \tag{2}$$

$D(\mathcal{M})$ being any domain of smooth functions, like $C_0^\infty(\mathcal{M})$ or $C^\infty(\mathcal{M})$. $\Delta := \nabla_a \nabla^a$ denotes the Laplace-Beltrami operator and ∇ means the covariant derivative associated to the metric connection. $d\mu_g$ denotes the natural Borel measure induced by the metric, and V is a *real* function of $C^\infty(\mathcal{M})$. (See [Mo99a, Mo99b, Mo99c] for discussions concerning the existence and the relevance of self-adjoint extensions of A_0 in $L^2(\mathcal{M}, d\mu_g)$ in both cases.) Throughout the text, if $(\mathcal{U}, \vec{x})$ is a local chart of the differentiable structure of a n-dimensional manifold $\mathcal{M}$ and thus $\vec{x} : \mathcal{U} \to \mathcal{V} : p \mapsto (x^1, \ldots, x^n)(p) = \vec{x}(p)$, $\mathcal{V} \subset I\!\!R^n$, we shall indentify $\mathcal{U}$ with $\mathcal{V}$, writing $(x^1, \ldots, x^n) \in \mathcal{U}$ as well as $p \in \mathcal{V}$, whenever it does not give rise to misunderstandings.

The heat-kernel coefficients for the Riemannian case and the Seeley-deWitt coefficients for the Lorentzian case, barring numerical factors, coincide with the coefficients which appear in the singular part of the Hadamard local solution (or Hadamard parametrix) for the linear homogeneous equation associated to the operator A_0 [Ch84, Ca90, Ga64, Fu91, BD82, Wa94] (see also [Mo99a, Mo99b, Mo99c] where the same notations used here are employed, for further references and comments.) The heat-kernel/Seeley-deWitt coefficients are given by the following definition (See [Mo99c] for further comments and remarks and for the corresponding differential recursive definition).

Definition 1.1. *Within the hypotheses on $\mathcal{M}$ and A_0 given above, in any fixed open geodesically convex neighborhood $\mathcal{N} \subset \mathcal{M}$, both the heat-kernel (for the Riemannian case) and Seeley-de Witt (for the Lorentzian case) coefficients are the functions defined on $\mathcal{N} \times \mathcal{N}$*

[1] In the *gr-qc* version of [Mo99c], we also assumed the positivity of A_0 in the Riemannian case and the geodesic completeness in general. Actually, these requirements are not necessary to assure the symmetry of the heat-kernel coefficients and they can be dropped as can be shown with a little modification of **Theorem 2.1** in [Mo99c].

and labeled by $j \in I\!N$*,*

$$a_0(x,y) = \Delta^{1/2}_{VVM}(x,y)\,, \tag{3}$$

$$a_{(j+1)}(x,y) = -\Delta^{1/2}_{VVM}(x,y)\int_0^1 \lambda^j \left[\Delta^{-1/2}_{VVM} A_{0x(\lambda)} a_j\right](x(\lambda),y)d\lambda\,. \tag{4}$$

$\lambda \mapsto x(\lambda)$ *is the unique geodesic segment from* $y \equiv x(0)$ *to* $x \equiv x(1)$ *contained completely in* $\mathcal{N}$*.*

Remark. This definition can be given as it stands also in the general case of a *non-singular semi-Riemannian* metric, namely, when more than one eigenvalue of the metric is negative and no eigenvalue vanishes. This is a straightforward consequence of the theory developed in **2.3** below.

$\Delta_{VVM}(x,y)$ is a (smooth or analytic[2] depending on the hypotheses on the metric) *bi-scalar* called the *van Vleck-Morette determinant* (see [Mo99c] for details). In any coordinate system $\vec{u} = (u^1,\cdots,u^D)$ defined in any open totally normal (or geodesically convex) neighborhood $\mathcal{N}$, if $x,y \in \mathcal{N}$ and $g := det g_{ab}$, we have

$$\Delta_{VVM}(x,y) := \left[(-1)^D \frac{g(\vec{x})}{|g(\vec{x})|}\right] \frac{1}{\sqrt{g(\vec{x})g(\vec{y})}} \det\left(\frac{\partial^2\sigma(\vec{x},\vec{y})}{\partial x^a \partial y^b}\right) > 0\,. \tag{5}$$

Above $x \equiv \vec{x}$, $y \equiv \vec{y}$ and $\sigma(\vec{x},\vec{y})$ is one half the "squared geodesical distance" of x to y (see [Mo99c] for details). The right-hand side of (5) is positive with the choice done for the first (constant) coefficient, not depending on the (fixed) *non-singular semi-Riemannian* signature of the metric (in particular, Riemannian or Lorentzian) and the used coordinates.

Remarks.
(1) These definitions can be given also if, in any non-singular semi-Riemannian case, $\mathcal{M}$ denotes a manifold with (smooth) boundary $\partial\mathcal{M}$. In this case it is also required that the fixed open geodesically convex neighborhood $\mathcal{N}$ does not intersect $\partial\mathcal{M}$. The results obtained in this paper can be straightforwardly generalized to manifolds with boundary.
(2) Differently from **Definition 1.1** in [Mo99c], here we prefer to distinguish explicitly between the Lorentzian and the Riemannian case employing a different nomenclature (heat-kernel or Seeley-deWitt coefficients respectively).
(3) The coefficients defined by (4) are either smooth if both the metric and V are smooth or (real) analytic if both the metric and V are (real) analytic (see [Mo99c]).

[2]Throughout this work "smooth" means C^∞ and "analytic" (C^ω) means *holomorphic* whenever the considered functions are complex valued.

These coefficients have been showed to be symmetric in x and y whenever the metric and V are smooth (or analytic) and the metric is *Riemannian* [Mo99c]. This holds true despite the non-symmetric definition (4) and despite several subtleties in the convergence properties of the off-diagonal heat-kernel expansion which could be non-asymptotic. As we said previously, this result is physically relevant within the theory of the point-splitting renormalization of the stress-energy tensor in curved spacetime concerning so-called Hadamard quasi-free quantum states. Indeed, the symmetry of the heat-kernel or Seeley-deWitt coefficients trivially implies the symmetry of the coefficients which appear in the *singular part* of (Euclidean or Lorentzian) Hadamard parametrix (see **1.3** of [Mo99c] for further details). The symmetry property is a sufficient[3] condition which assures a final well-behaved renormalized (Euclidean or Lorentzian) stress-energy tensor (see [Wa78, FSW78] and references in [Wa94]). Such important requirement has been assumed in the mathematical-physics literature without an explicit proof to the knowledge of the author (see [Mo99c] for further comments). This paper is devoted to show that the symmetry holds true also in the Lorentzian case which is much more interesting on physical grounds.

1.2. *The rough idea of the proof of the symmetry for the Lorentzian case.* In principle, a direct attempt to prove the symmetry could be performed as we done in the Riemannian case [Mo99c]. That is, by employing the so-called Seeley-deWitt (or Schwinger-deWitt) expansion of the integral kernel associated to the one-parameter group of unitary operators generated by some suitable self-adjoint extension of A_0 [BD82, Fu91, Ca90, Mo99c]. In fact, the Seeley-deWitt coefficients are just *the* coefficients of this expansion. This expansion is the direct analogue of the heat-kernel expansion [Ch84, Mo99c]. Anyway, the convergence properties of the former are much more complicated than those of the latter (see discussions and references in [Fu91, Mo99c]), so we prefer to follow an alternative way, which seems to be more interesting also on mathematical-physics grounds. The rough idea of the proof involves a sort of *"local" Wick rotation*, namely a somehow continuation of the relevant coefficients from the Lorentzian theory into the Riemannian one. By the uniqueness theorem of the analytical continuation, this should entail the generalization of the symmetry to the Lorentzian coefficients from the symmetry of Riemannian coefficients. We shall see that a sort of "local" Wick rotation can be performed, not depending on the presence of time-like Killing vectors, provided the metric and the potential V are (real) analytic functions of the coordinates. Finally, the generalization of the symmetry to the smooth *non-analytic* case can be obtained exactly as we done in the Riemannian case, making use of **Proposition 2.1** in [Mo99c]. This is, by an approximation of smooth metrics by analytic metrics and smooth functions V by analytic functions. The intriguing issue is the generalization of the Wick rotation from Minkowski spacetime

[3] It is not so clear whether or not this condition is necessary. After the appearance of the first version of [Mo99c], R.M. Wald pointed out to me that a weaker requirement should be, in practice, sufficient (see comment before **Proposition 2.1** in [Mo99c]).

to curved *non-stationary* spacetimes. This is the argument of the next section.

2 "Local Wick rotation".

2.1. *A generalized local Wick rotation.* In QFT in flat spacetime the so-called (spatial) *Wick rotation* is a useful tool in spite of quite a vague definition. Roughly speaking, the Wick rotation is nothing but an analytical continuation of the Minkowskian time coordinate into imaginary values: $t \to i\tau$ for $\tau \in I\!R$. This is done in order to produce a Riemannian background where one can define the Euclidean QFT. Formally, the metric changes as follows

$$\Phi_L = -dt \otimes dt + \sum_{i=1}^{d} dx^i \otimes dx^i \quad \to \quad \Phi_R = d\tau \otimes d\tau + \sum_{i=1}^{d} dx^i \otimes dx^i \,. \tag{6}$$

Such a procedure is performed for several goals, e.g., to make sensible the path integral as a Wiener measure or to build up the thermal QFT. In *curved* spacetime, the use of the Wick rotation is much more problematic. In particular, there is no guarantee for the fact that the continued Riemannian metric is *real* whenever the initial *Lorentzian* metric is real. In principle, a somehow sufficient condition which gives a real Riemannian metric is given by the requirement of a *static metric* [Wa79, FR87, Fu91, Wa94]. If the metric is only *stationary* the Wick rotation is more problematic and generally involves the analytic continuation of further parameters than the time coordinate [Ha77]. In spite of these difficulties, the Wick rotation is successfully used in QFT in curved spacetime, Quantum Gravity and black holes theory, where it is a very powerful tool in studying black hole thermodynamics in particular [GH93].
In this work, to get the proof of the symmetry of Seely-deWitt coefficients, we want to generalize the Minkowskian Wick rotation to quite a general Lorentzian manifold dropping any hypothesis concerning the presence of temporal Killing fields. As far as we are concerned, a sort of Wick rotation of the *metric* is sufficient to prove the first step of our symmetry theorem. To this end, let us focus our attention on (6) once again. Notice that the same result can be obtained by an analytic continuation of the *metric* rather than the time coordinate. In fact, we are free to interpret (6) as

$$\Phi_L = g_{Lab} dx^a \otimes dx^b \quad \to \quad \Phi_R = g_{Rab} dx^a \otimes dx^b \,. \tag{7}$$

where

$$g_{Lab} = diag(-1, 1, 1, 1) \quad \text{and} \quad g_{Mab} = diag(1, 1, 1, 1)\,.$$

Differently from the customary interpretation, the metric has now changed, since the eigenvalue -1 has been continued into a final eigenvalue $+1$, but the manifold has remained the initial one. The changes have taken place only in each *(co-)tangent fiber*. In

this sense the continuation is "local". In principle, such a procedure could be used also in curved spacetime without the requirement of a static or stationary metric. We are not interested in the issue about which properties of the customary interpretation are preserved by the new interpretation. Only two important points have to be remarked: Following the new procedure, there is no guarantee for the fact that the Riemannian metric so obtained is a vacuum solution of Euclidean Einstein equation if the initial Lorentzian metric is a vacuum solution of the Lorentzian Einstein equations; moreover it is worthwhile stressing that the found procedure is very non-unique.
The use of complex metrics is compulsory if one assume the absence of pathologies in the structure of geodesically convex neighborhoods during the continuation procedure. Such an absence is essential as far as our main goal, i.e., the proof of symmetry of Hadamard coefficients, is concerned, because these coefficients are defined just in geodesically convex neighborhoods. If we want to pass from a Lorentzian to a Riemannian metric continuously, the signature has to change from $(-,+,\cdots,+)$ to $(+,+,\cdots,+)$. Therefore, the determinant of the metric, in any fixed coordinate frame, must change sign somewhere. This implies that, employing only *real* metrics, some of these must become *singular* somewhere during the continuation. Therefore, pathologies would arise concerning the exponential maps and the structure of geodesical neighborhoods. On the other hand, the use of *complex* coefficients of the metric makes sense only in complex manifolds. For instance, in general, the equations of the geodesics admit no solution in *real* coordinates with *complex* coefficients of the metric. Therefore, we are forced to extend the initial Lorentzian metric and manifold to complex values in complex coordinates. A natural way to do this follows from the assumption of an initial *real analytic* metric. The "complex metrics" which arise in the continuation procedure are not Hermitian or Kählerian but generalize the concept of a pseudo-Riemannian metric into a complex context.

To proceed with our idea of a local generalized Wick rotation we need a well-known preliminary definition.

Definition 2.1. *Let $(\mathcal{M}, \mathbf{g})$ be a C^k ($k \in \{2, \ldots, \infty, \omega\}$) Lorentzian ($D = d+1$)-dimensional manifold. Choose any embedded spacelike hypersurface $\mathcal{S}$ and an open neighborhood $\mathcal{O}$ such that $\mathcal{O} \cap \mathcal{S} \neq \emptyset$. Taking $\mathcal{O}$ sufficiently small (preserving the condition above) if necessary, any admissible (C^k) local coordinate system defined in $\mathcal{O}$, $\vec{x} = (x^0, \ldots, x^d)$ with, $(x^1, \ldots, x^d) \in \Omega$ open subset of $I\!R^D$ and $x^0 \in\,]-\delta, \delta[\,$, $\delta > 0$, such that, in $\mathcal{O}$,*

$$\begin{aligned} \mathcal{S} \cap \mathcal{O} &= \{(x^0, \ldots, x^n) \in\,]-\delta, \delta[\, \times \Omega \mid x^0 = 0\}\,, &(8)\\ g_{00} &= -1\,, &(9)\\ g_{0a} &= 0 \quad \textit{for} \quad a > 0\,. &(10)\end{aligned}$$

is said **local synchronous coordinate system with respect to $\mathcal{S}$.**

Concerning the existence of such coordinate systems, see [Wa84] and, for a more general mathematical discussion, see Chap.7 of [ON83] where these coordinates are called "Fermi coordinates with respect to a given spacelike hypersurface". A sketch of the proof of their existence will also be given within the proof of **Theorem 2.2**.

Remark. Any point $p \in \mathcal{M}$ belongs to an embedded spacelike hypersurface: Such a hypersuface can be obtained as $\mathcal{S}_p = \{exp_p(X^a \mathbf{e}_a) \mid X^0 = 0 \ (X^1, \ldots, X^d) \in \Omega\}$, where Ω is a suitable small neighborhood of the origin of $I\!R^d$, $(\mathbf{e}_0, \ldots, \mathbf{e}_d)$ being an orthonormal base of $T_p(\mathcal{M})$ with $(\mathbf{e}_0, \mathbf{e}_0) = -1$.
Notice that the found hypersurface about p is not uniquely defined.

Let us consider an analytic manifold $\mathcal{M}$ endowed with an analytic Lorentzian metric $\mathbf{g}$. Take a complex analytic continuation of a synchronous coordinate system defined in a open neighborhood $\mathcal{O}$ about a point $p \in \mathcal{M}$, $\vec{z} = (z^0, z^1, \cdots, z^d)$ with $z^a = x^a + iy^a$ into a complex open neighborhood $\mathcal{G} \in \mathbb{C}^D$ (containing the initial real domain of definition of the coordinates). Suppose that analytic continuations of the the functions $\vec{z} \mapsto g_{ab}(\vec{z})$ are defined in $\mathcal{G}$. In general $\vec{z} \mapsto g_{ab}(\vec{z})$ are complex-valued functions but preserve (9) and (10). By these functions it is possible to define a *non-singular* "complex pseudo-metric" (the rigorous definition will be given in **Definition 2.2**) $\vec{z} \mapsto \mathbf{g}(\vec{z}) = g_{ab}(\vec{z})\, dz^a \otimes dz^b$ on $\mathcal{G}$. Finally, fix an arbitrary real $\lambda > 0$ and consider the class of "complex pseudo-metrics" $\left\{\mathbf{g}_{(\lambda\theta)}\right\}_\theta$ where $\theta \in \mathbb{C}$, defined in the coordinates of $\mathcal{G}$ by

$$g_{(\lambda\theta)00}(\vec{z}) \;:=\; g_{00}(\vec{z})\lambda^{2\theta/\pi}e^{i\theta}\,, \tag{11}$$
$$g_{(\lambda\theta)ab}(\vec{z}) \;:=\; g_{ab}(\vec{z}) \quad \text{for} \quad (a,b) \neq (0,0)\,. \tag{12}$$

We want to use this class to continue the initial *Lorentzian* metric obtained for $\theta = 0$, $\mathbf{g} = \mathbf{g}_{(\lambda 0)}$ into a final *Riemannian* "Wick-rotated metric", obtained for $\theta = \pi$. Indeed, within our hypotheses the "Wick-rotated metric" $\bar{\mathbf{g}}_\lambda(\vec{z}) := \mathbf{g}_{(\lambda\pi)}(\vec{z})$ defines a *real*, *non-singular* and *Riemannian* metric for $\vec{z} = \vec{x} \in \mathcal{O}$ when it acts on *real* (with respect to the considered coordinates) vectors. In particular, fixed any positive real λ, $\mathbf{g}_{(\lambda\theta)}(\vec{z})$ is *non-singular* in a complex open neighborhood of $[0,\pi] \times \mathcal{O}$. More strongly, in a sense we shall specify later, fixed the parameter λ, the procedure *preserves geodesically convex neighborhoods* for complex value of θ. In practice, the presented procedure locally defines an *analytic* continuation of metrics[4] which interpolates through complex metrics, from *Lorentzian* to *Riemannian* metrics and preserves the local geodesical structure at each step. As we shall see, the apparently superfluous parameter λ plays a central rôle in using the local Wick rotation to get the symmetry of Seeley-deWitt coefficients.

[4] Notice that the functions $(\theta, \vec{z}) \mapsto g_{(\lambda\theta)ab}(\vec{z})$ are indeed *analytic* in (θ, z).

Let us state some of the results argued above into a precise theorem.

Theorem 2.1. *Let $(\mathcal{M}, \mathbf{g})$ be a $(D = d+1)$-dimensional Lorentzian manifold with class C^ω. Let $\mathcal{O} \subset \mathcal{M}$ be any open set endowed with local synchronous coordinates (with respect to some embedded hypersurface) $\vec{x} = (x^0, \cdots, x^d)$ and consider the coefficients of the metric $g_{ab}(\vec{x})$ in these coordinates. Fix a real $\lambda > 0$.*
Then, there is a complex open set $\mathcal{G} \subset \mathbb{C}^D$ endowed with a differentiable structure induced by coordinates $\vec{z} = (z^0, \cdots, z^d)$ with $z^a = x^a + iy^a$, $a = 0, \cdots, d$ and $\mathcal{O} \subset \mathcal{G}$ (in the obvious sense), where the components of the metric $\vec{x} \mapsto g_{ab}(\vec{x})$, $a, b = 0, \cdots, d$, can be analytically continued into analytic complex functions $\vec{z} \mapsto g_{ab}(\vec{z})$. Moreover the functions defined in $\mathbb{C} \times \mathcal{G}$, $\vec{z} \mapsto g_{(\lambda\theta)ab}(\vec{z})$, where $g_{(\lambda\theta)ab}(\vec{z})$ have been defined in (11) *and* (12)*, define a θ-parametrized class $\{\mathbf{g}_{(\lambda\theta)}\}_\theta$, $\theta \in \mathbb{C}$, of complex analytic $(0,2)$-degree symmetric fields*

$$\mathbf{g}_{(\lambda\theta)}(z) := g_{(\lambda\theta)ab}(\vec{z})\, dz^a \otimes dz^b \,, \tag{13}$$

$(z \equiv \vec{z})$ which are non-degenerate everywhere in the complex manifold $\mathcal{G}$.
In $\mathcal{O}$, this class analytically continues in the parameter θ, the initial Lorentzian metric $\mathbf{g} = \mathbf{g}_{(\lambda 0)}$ into the Riemannian **Wick-rotated metric**

$$\bar{\mathbf{g}}_\lambda := \mathbf{g}_{(\lambda\pi)} \,. \tag{14}$$

Proof. It is straightforward if one takes into account that the components of $g_{(\lambda\theta)ab}(\vec{z})$ with $a, b > 0$ do not depend on θ and λ and also noticing that (9) and (10) hold true for the "complex-continued metrics" in (13). Hence,

$$|\det\{[g_{(\lambda\theta)ab}(\vec{z})]_{a,b=0,\dots,d}\}| = |e^{\theta\,[i+(2/\pi)\ln\lambda]}|\, |\det\{[g_{ab}(\vec{z})]_{a,b=1,\dots,d}\}| \,.$$

The former factor on the right-hand side is positive and cannot vanish if $\lambda > 0$ whatever $\theta \in \mathbb{C}$ and the latter factor defines a continuous function of the only variable $\vec{z} \in \mathcal{G}$ which preserves the positive sign in a open complex neighborhood of each point $\vec{z} = \vec{x} \in \mathcal{O}$ where the function is positive by hypotheses. We can redefine $\mathcal{G}$ as the union of all of these open neighborhoods. $\Box$

Remark. Notice that we have used a little misuse of notations in the last statement of the theorem. Indeed, the initial Lorentzian metric $\mathbf{g}(x) = g_{ab}(\vec{x})\, dx^a \otimes dx^b$ and the Wick-rotated Riemannian metric $\bar{\mathbf{g}}_\lambda(x) = \bar{g}_{(\lambda)ab}(\vec{x})\, dx^a \otimes dx^b$ are defined in the *real*, with respect to the base induced by the coordinates specified above $z^a = x^a + iy^a$, cotangent space of $\mathcal{O}$. Conversely, all interpolating fields (including those corresponding to the values $\theta = 0, \pi$) $\mathbf{g}_{(\lambda\theta)}$ in (13) are defined in the whole *complex* cotangent space of $\mathcal{O}$. Anyway, we shall use these notations also in the following since it does not produces misunderstandings.

Note. The local Wick rotation we have defined above can be generalized to a more large class of coordinates with a precise physical meaning, namely, coordinates where x^0 represents a "true" time and $x^1, \ldots, x^d$ represent "true" spatial coordinates. In other words, in these coordinates, it must hold $g_{00} < 0$ as well as $g^{00} < 0$. Within this general approach, fixing a point $p \in \mathcal{M}$, the parameter λ is related to the relative velocity between the rest reference d-dimensional space (subspace of $T_p(\mathcal{M})$) of the infinitesimal observer evolving along $\partial_{x^0}|_p$ and the d-dimensional reference space (subspace of $T_p(\mathcal{M})$) "normal" to the vector $dx^0|_p$. Also **Theorem 2.1** and **Theorem 2.4** below can be generalized for these "physical" coordinates but the proofs are much more complicate. [5]. We also have the following almost straightforward result, which is interesting by its own not depending on our final goal. It shows that the local procedure defined above can be relatively globalized about any spacelike embedded hypersurface (e.g. a Cauchy surface if $\mathcal{M}$ is globally hyperbolic.) also when *global* synchronous coordinates with respect to it do not exist. Anyway, we stress that we shall use the local result only in the proof of symmetry theorem.

Theorem 2.2. *Let* $(\mathcal{M}, \mathbf{g})$ *be a* $(D = d+1)$*-dimensional Lorentzian manifold with class* C^ω *and time-oriented. Fix any real* $\lambda > 0$*. Let* $\mathcal{S} \subset \mathcal{M}$ *any fixed embedded space-like hypersurface with class* C^ω*.*
Then, there is an open D*-dimensional open Lorentzian sub-manifold of* $\mathcal{M}$*,* $\mathcal{N}$ *containing* $\mathcal{S}$ *which admits the class* $\mathcal{A}(\mathcal{N})$ *of (*C^ω*)* time-oriented local synchronous coordinates with respect to $\mathcal{S}$ *as an atlas. Moreover, the Wick-rotated metrics defined by* (14) *in each local coordinate system of* $\mathcal{A}(\mathcal{N})$ *induce a* global *Riemannian* C^ω *metric on the whole sub-manifold* $\mathcal{N}$*.*

Sketch of Proof. See the **Appendix** $\square$

In the next part we shall consider some features of the Wick-rotated metrics. To this end we need some definitions and results concerning complex metrics in complex analytical manifolds.

2.3. *Complex pseudo-Riemannian manifolds.* To give a precise status to the complex field $\mathbf{g}_{(\lambda\theta)}$ defined on the complex manifold $\mathcal{G}$ presented in **Theorem 2.1**, we introduce the concept of a *complex pseudo-Riemannian manifold* and a *complex pseudo-Riemannian metric*. Also with different nomenclature, several results obtained in the following can be found in [LB83] (see also [Cs96]). First of all, we give some results concerning the existence and the analyticity of the exponential map in complex manifolds with a generally complex pseudo-Riemannian metric. Afterwards we discuss the existence of geodesically convex neighborhoods and related features. Let us start by giving the definition of a

[5] A general discussion on these arguments, and all the relevant proofs, can be found in the first *gr-qc* version of this paper.

complex pseudo-Riemannian manifold.

Definition 2.2. *A complex analytic manifold* $\mathcal{M}$ ([KN63]) *endowed with an analytic non-degenerate* $(0,2)$*-degree symmetric tensorial field* $\mathbf{g}$ *is said* **complex pseudo-Riemannian manifold** *and the field* $\mathbf{g}$ *is said* **complex pseudo-metric**. *The complex pseudo-metric induces a* non-degenerate *complex quadratic form* $\mathbf{V} \mapsto \mathbf{g}(z)(\mathbf{V}, \mathbf{V})$*, in the tangent space* $T_z(\mathcal{M})$ *at any point* $z \in \mathcal{M}$*. We call such a quadratic form the* **complex pseudo scalar product** *induced in* $T_z(\mathcal{M})$ *by the complex pseudo-metric.*

Remarks.
(1) It is worthwhile stressing that the complex pseudo scalar product induced on the tangent spaces is *not* Hermitian and the metric is *not* Kählerian.
(2) It is clear that the manifold $\mathcal{G}$ introduced in **Theorem 2.1**, endowed with any fixed field $\mathbf{g}_{(\lambda\theta)}$, is a complex pseudo-Riemannian manifold.
(3) The equations of the geodesics take the usual formula with the difference that the connection coefficients of the Levi-Civita connection (see below) induced by the complex pseudo-metric are complex analytic functions of the considered coordinates.

Concerning the equation of the geodesics we can apply the following general lemma.

Lemma 2.1. *Let* $f : (z, Y, \alpha) \mapsto f(z, Y, \alpha) \in \mathbb{C}^n$ *be a function in* $C^\omega(\bar{\mathcal{C}}; \mathbb{C}^n)$ [6]*, with* $\mathcal{C} = B_{r_1}(z_0) \times B_{r_2}(y_0) \times B_{r_3}(\alpha_0)$ *where* $B_{r_1}(z_0), B_{r_3}(\alpha_0) \subset \mathbb{C}$ *and* $B_{r_2}(y_0) \subset \mathbb{C}^n$ *are open balls with radii* $r_1, r_2, r_3 > 0$ *centered in* z_0, y_0, α_0 *respectively. Consider the differential equation system depending on the parameter* $\alpha \in \bar{B}_{r_3}(\alpha_0)$

$$\frac{dY}{dz} = f(z, Y, \alpha) \qquad Y \in C^1(B_{r_1'}(z_0); \mathbb{C}^n) \quad \text{for some} \quad r_1' > 0,\ r_1' < r_1 \tag{15}$$

and initial condition

$$Y(t_0) = \bar{y}_0, \qquad \bar{y}_0 \in \bar{B}_{r_2'}(y_0), \quad \text{where } r_2' > 0, \quad \text{is fixed and} \quad r_2' < r_2\,. \tag{16}$$

(a) *A solution of Eq.* (15) *with initial condition* (16) *exists and is unique in any set* $\bar{B}_{r_1'}(z_0)$*, provided that*

$$0 < r_1' < Min\,(r_1, \delta', \delta'')\ , \tag{17}$$

where

$$\delta' = (r_2 - r_2')/Sup\left\{||f(z, y, \alpha)||\ |\ (z, y, \alpha) \in \bar{\mathcal{C}}\right\}\ ,$$

[6] Notice that $\bar{\mathcal{C}}$ is closed. We say that f is analytic in a *closed* set when it is possible to continue f into an analytic function defined in a *open* set which includes the closed set.

and

$$\delta'' = 1/Sup\left\{2\sqrt{n\,Tr\nabla f^*(t,y,\alpha)^T\nabla f(t,y,\alpha)}\ \mid\ (z,y,\alpha)\in\bar{\mathcal{C}}\right\}\ .$$

(b) *This solution satisfies* $Y(z)\in\bar{B}_{r_2}(y_0)$ *for any* $z\in\bar{B}_{r'_1}(z_0)$*, whatever* $\bar{y}_0\in\bar{B}_{r'_2}(y_0)$ *and* $\alpha\in\bar{B}_{r_3}(\alpha_0)$*.*

(c) *Moreover, varying also* y_0 *and* α*, and writing down the dependence on these variables explicitly, the function* $(t,\bar{y}_0,\alpha)\mapsto Y(t,\bar{y}_0,\alpha)$ *is analytic. In particular, it belongs to* $C^\omega(\bar{B}_{r'_1}(z_0)\times\bar{B}_{r'_2}(y_0)\times\bar{B}_{r'_3}(\alpha_0))$ *for any* $r'_3>0$ *with* $r'_3<r_3$*.*

Proof. See the **Appendix**.

We can apply the lemma above to the equations of the geodesics for a complex pseudometric $\mathbf{g}$ in coordinates $\vec{z}=(z^1,\cdots,z^D)$. For the moment we do not consider the further parameter α. The *first-order* geodesical equation system reads, for the complex pseudometric $\mathbf{g}$ in the coordinates $\vec{z}=(z^1,\cdots,z^D)$,

$$\frac{dz^a(t,\vec{y},\mathbf{V})}{dt} = U^a(t,\vec{y},\mathbf{V}) \tag{18}$$

$$\frac{dU^a(t,\vec{y},\mathbf{V})}{dt} = -\Gamma^a_{bc}(\vec{y})U^b(t,\vec{y},\mathbf{V})U^c(t,\vec{y},\mathbf{V})\,, \tag{19}$$

for $a=1,\cdots,D$. (The sum over the repeated indices is understood). Above, the complex Levi-Civita connection coefficients are defined, as usual, by

$$\Gamma^a_{bc}(\vec{z}) := \frac{1}{2}g^{ad}(\vec{z})\left(\frac{\partial g_{db}(\vec{z})}{\partial z^c}+\frac{\partial g_{cd}(\vec{z})}{\partial z^b}-\frac{\partial g_{bc}(\vec{z})}{\partial z^d}\right)\,, \tag{20}$$

$\vec{y}$ and $\mathbf{V}$ are, respectively, the initial position and the initial velocity of the geodesic segment evaluated at $t=0$. The equations (18) and (19) for $t\in\mathbb{C}$, locally admit a unique solution which satisfies the given initial conditions. The existence of sets where the solution exists, is unique and is analytic is assured by **Lemma 2.1**. Let us indicate the local solution of the system above by

$$t\mapsto\gamma(t,\vec{z},\mathbf{V})\,, \tag{21}$$

for $t\in B_\delta(0)$, $\delta>0$, $(\vec{y},\mathbf{V})\in B_\rho(\vec{y}_0)\times B_r(\mathbf{0})$, $\rho,r>0$. Exactly as in the real Riemannian case, for any fixed complex number $c\neq 0$, (18) and (19) entail the identity

$$\gamma(ct,\vec{z},\mathbf{V}/c)=\gamma(t,\vec{z},\mathbf{V})\,. \tag{22}$$

This means that if, for instance, $2>\delta>0$, passing to the new variable $t'=(2/\delta)t$, we can work with geodesics defined in the interval $t'\in B_2(\vec{0})$ provided r is replaced

by $r' = (\delta/2)r < r$. This can be done preserving all remaining properties concerning the analyticity and not depending on the initial condition $\vec{z}$ (and not depending on any further parameter α as that in **Lemma 2.1**). Since there is no ambiguity we can use the name r instead of r' and t instead of t'. Therefore, from now on, we suppose $t \in B_2(\vec{0})$. With this choice, a maps (21) with the restriction $t \in \bar{B}_1(\vec{0})$ will be called *complex geodesic segment.* It is worthwhile stressing that this is not a "usual" segment because the parameter t corresponds to *two* real parameters.
A complex geodesic segment restricted to the real axis in the domain, $s \in \mathbb{R}$,

$$s \mapsto \gamma(s, \vec{z}, \mathbf{V}) \,, \quad \text{where} \quad s \in [0, 1] \tag{23}$$

will be called *real-parameter geodesic segment.* Notice that it satisfies (18) and (19) in the variable s and is real analytic in this variable. It determines the whole complex geodesical segment by analytic continuation. Obviously, these definitions does not depend on the chosen coordinates, so sometimes we shall use, e.g., p instead of $\vec{y}$ in the second argument of a geodesic segment.
The *exponential map* is, as usual, given as the analytic map, defined in an opportune *open* set

$$E = \bigcup_{p \in \mathcal{M}} \{p\} \times E_p \subset T(\mathcal{M}) \,, \tag{24}$$

where E_p is an open neighborhood of the origin of $T_p(\mathcal{M})$ we shall specify shortly,

$$exp : E \to \mathcal{M} : (p, \mathbf{V}) \mapsto \gamma(1, p, \mathbf{V}) \,. \tag{25}$$

The exponential map *centered in* $p \in \mathcal{M}$ is the map

$$exp_p : E_p \to \mathcal{M} : \mathbf{V} \mapsto \gamma(1, p, \mathbf{V}) \,. \tag{26}$$

Obviously, these definitions does not depend on the used coordinates and, changing the domains one finds restrictions or extensions of the same function. Since, for any point $p \in \mathcal{M}$, it holds

$$d(exp_p)_{\mathbf{0}}\mathbf{V} = \frac{d}{dt}|_{t=0} exp_p(t\mathbf{V}) = \frac{d}{dt}|_{t=0}\gamma(1, p, t\mathbf{V}) = \frac{d}{dt}|_{t=0}\gamma(t, p, \mathbf{V}) = \mathbf{V} \,,$$

there is a open neighborhood, which we can assume to be *starshaped*, of the origin of the tangent space at p, where the exponential map centered in p defines an analytic diffeomorphism onto a neighborhood of p in the manifold. With an open starshaped neighborhood of $\mathbf{0}$ we mean an open neighborhood of $\mathbf{0}$ such that if $\mathbf{V}$ belongs to this neighborhood, also any $\lambda\mathbf{V}$, with $\lambda \in \mathbb{R}$ and $0 \le \lambda \le 1$, belongs to the neighborhood[7]. For instance, any complex open ball of centered in $z \in \mathbb{C}$ with positive radius $r > |z|$ is

[7] It is possible to give a stronger definition requiring $\lambda \in \mathbb{C}$ and $|\lambda| \le 1$. Anyway, throughout this paper we use the weaker definition.

an open starshaped neighborhood of the origin of $\mathbb{C}$. We can take each E_p above as fixed open starshaped neighborhoods of $\mathbf{0}$ where the exponential map centered on p define a diffeomorphism. Working in fixed local coordinates, it is trivially possible to choose such sets E_p such that E given in (24) is also open. Hence, the map

$$\phi : (p, \mathbf{X}) \mapsto (p, exp\mathbf{X}) \ , \tag{27}$$

defines a diffeomorphism in E onto $\phi(E) \subset \mathcal{M} \times \mathcal{M}$ because it is injective and its differential does not vanish in each point of the domain.

As usual, a *normal* neighborhood of the point $p \in \mathcal{M}$, is an open neighborhood of p with the form $\mathcal{N}_p = exp_p(S)$ whenever $S \subset E_p \subset T_p(\mathcal{M})$ is an open starshaped neighborhood of the origin of $T_p(\mathcal{M})$ where the exponential map centered in p defines an analytic diffeomorphism. Then, the components of the vectors $\mathbf{V} \in T_p(\mathcal{M})$, with respects to a fixed base, contained in S, define *normal coordinates* on $\mathcal{M}$ centered in p via the function $\mathbf{V} \mapsto exp_p\mathbf{V}$. Notice that any $q \in \mathcal{N}_p$, due to (22) and the starshapedness of $exp_p^{-1}(\mathcal{N})_{\surd}$, can be connected with p by only one complex geodesic segment "starting from p" at $t = 0$ and "terminating in q" at $t = 1$, such that the associated real-parameter geodesic segment is completely contained in $\mathcal{N}_p$. (Using the stronger definition of starshaped neighborhood suggested in the previous footnote, the whole complex segment geodesic would be contained in $\mathcal{N}_p$.) Finally, in normal coordinates centered in p, due to (22), the equation of a complex geodesic which starts form p is a linear function of the parameter. This involves that the connection coefficients vanishes at p if evaluated in these coordinates.

Similarly to the real case, we define a *totally normal neighborhood* of a point $p \in \mathcal{M}$ as a neighborhood [8] of p, $\mathcal{V}_p \subset \mathcal{M}$, such that, if $q \in \mathcal{V}_p$, there is a normal neighborhood of q, $\mathcal{N}_q$, with $\mathcal{V}_p \subset \mathcal{N}_q$. Therefore, if q and q' belong to the same totally normal neighborhood, there is only one complex geodesic segment which "connects" these two points (respectively for $t = 0$ and $t = 1$) such that the associated real-parameter geodesic segment is completely contained in normal neighborhoods centered in q and q' respectively, $\mathcal{N}_q$ and $\mathcal{N}_{q'}$.

Finally, a *complex geodesically convex* neighborhood of a point $p \in \mathcal{M}$ should be defined as a totally normal neighborhood of p, $\mathcal{U}_p$, such that, for any couple $q, q' \in \mathcal{U}_p$, there is only one complex geodesic segment which is completely contained in $\mathcal{U}_p$ and "connects" q (for $t = 0$) and q' (for $t = 1$). In fact, also in the simplest case of a complex manifold $\mathcal{M} \subset \mathbb{C}$ these neighborhoods do not exist barring trivial cases (e.g., $\mathcal{U}_p = \mathcal{M} = \mathbb{C}$). However, a weaker definition can successfully be given. A *geodesically linearly convex* neighborhood of a point $p \in \mathcal{M}$ is defined as a totally normal neighborhood of p, $\mathcal{U}_p$, such that, for any couple $q, q' \in \mathcal{U}_p$, there is only one *real-parameter* geodesic segment which is completely contained in $\mathcal{U}_p$ and connects q (for $s = 0$) and q' (for $s = 1$).

It is not so obvious, at this point, that our complex pseudo-Riemannian structure admits

[8] In this work, a neighborhood of a point is any set which includes an open set which contains the point.

totally normal and geodesically linearly convex neighborhoods. Actually this is the case.

Theorem 2.3. *Let* $(\mathcal{M}, \mathbf{g})$ *be a complex pseudo-Riemannian manifold. For each point* $p \in \mathcal{M}$ *there is a local base of the topology* $\{\mathcal{G}_{pj}\}_{j\in I\!R}$ *consisting of open totally normal, geodesically linearly convex neighborhoods of the point* p*. Moreover each* $\bar{\mathcal{G}}_{pj}$ *is also totally normal and geodesically linearly convex for any* $j \in I\!R$ *and* $\bar{\mathcal{G}}_{pj} \subset \mathcal{G}_{pj'}$ *if* $j < j'$*.*

Proof. See the **Appendix**. □

Remark. The definition of linearly geodesically convex neighborhoods could not seem very natural. Anyway, it can be given in a more natural way for open sets (see also [LB83]). To this end, we leave to the reader the proof of the following relevant proposition.

Proposition 2.1. *Given a complex pseudo-Riemannian manifold* $(\mathcal{M}, \mathbf{g})$*, an* open *set* $\mathcal{U} \subset \mathcal{M}$ *is linearly geodesically convex, if and only if there is an open set* $E(\mathcal{U}) \subset T(\mathcal{M})$*, with*

$$E(\mathcal{U}) = \bigcup_{p\in\mathcal{U}} \{p\} \times E(\mathcal{U})_p \,, \tag{28}$$

$E(\mathcal{U})_p \subset E_p$ *being an open starshaped neighborhood of the origin of* $T_p(\mathcal{M})$*, such that the map*

$$\phi_{\mathcal{U}} : E(\mathcal{U}) \to \mathcal{U} \times \mathcal{U} : (p, \mathbf{X}) \mapsto (p, exp_p \mathbf{X}) \tag{29}$$

is an analytic diffeomorphism onto $\mathcal{U} \times \mathcal{U}$*.*

The existence of totally normal and linearly geodesically convex neighborhoods allow us to define the *one half squared complex pseudo-distance* or *complex world function* similarly to the case of real metrics. Given an open linearly geodesically convex neighborhood, or, more simply, an open totally normal neighborhood, $\mathcal{U}$, the *complex world function* is given by

$$\sigma(p, q) := \frac{1}{2}\mathbf{g}(p)(exp_p^{-1}(q), exp_p^{-1}(q)) \quad \text{for any} \quad q, p \in \mathcal{U} \,. \tag{30}$$

Since all functions involved on the right-hand side of (30) are analytic, it must hold $\sigma \in C^\omega(\mathcal{U}\times\mathcal{U})$. Moreover, essentially from the conservation of $\mathbf{g}(\mathbf{V}, \mathbf{V})$ along any complex geodesic segment due to the geodesical transport, $\mathbf{V}$ being the tangent vector, we have the following properties which generalize well-known Riemannian and Lorentzian results [Fu91].

$$\sigma(p, q) \;=\; \sigma(q, p) \,, \tag{31}$$

$$\sigma(p,q) = \frac{1}{2}\nabla_{(p)a}\sigma(p,q)\nabla^a_{(p)}\sigma(p,q)\,, \tag{32}$$
$$\nabla_{(p)}\sigma(p,q) = \frac{d}{dt}|_{t=1}\gamma(t,q,p)\,, \tag{33}$$

where the function on the right hand side of (33) is defined below. In an open geodesically linearly convex neighborhood or, more simply, in an open totally normal neighborhood, $\mathcal{U}$, we can define the function

$$\gamma(t,p,q) := \gamma(t,p,exp_p^{-1}(q)) \quad \text{for any} \quad q,p \in \mathcal{U} \quad \text{and} \quad t \in \bar{B}_1(0)\,, \tag{34}$$

which gives the complex geodesic segment "connecting" the point p ($t=0$) and the point q ($t=1$) as a function of the extreme points. Once again, trivially, $\gamma \in C^\omega(\bar{B}_1(0)\times\mathcal{U}\times\mathcal{U})$. The property (32) is a consequence of the property (33). The latter is not very simple to prove. A direct way is the following. Consider a *normal coordinates system* centered in q. In these coordinates, if $\vec{x} = \vec{x}(t)(= t\vec{x}(1))$ is the equation of the real-parameter geodesic segment from $q \equiv \vec{x}(0) = \vec{0}$ to $p \equiv \vec{x}(1)$, it holds trivially, since the integrand actually does not depend on t due to the parallel transport, $\sigma(p,q) = \frac{1}{2}\int_0^1 g_{ab}(x(t))\frac{dx^a}{dt}\frac{dx^b}{dt}dt$. Then we can vary the curve in the integrand within any family of (real-parameter segment) geodesics $\vec{x}_\alpha = \vec{x}_\alpha(t)$, $t \in]-\delta,\delta[$. Assume also that the dependence on α is smooth, $\vec{x}_0(t) := \vec{x}(t)$ and $\vec{x}_\alpha(0) = \vec{0}$ for any α. This defines a functional $\sigma = \sigma[\vec{x}_\alpha]$. Using the equation of the geodesic for $\alpha = 0$, it is quite trivial to get by integration by parts that $\frac{d\sigma[\vec{x}_\alpha]}{d\alpha}|_{\alpha=0} = g_{ab}(\vec{x}(1))x^a(1)\frac{dx^b_\alpha(1)}{d\alpha}|_{\alpha=0}$. On the other hand, since each curve of the family is a geodesic and $p \equiv \vec{x}(1)$, we have $\frac{d\sigma[\vec{x}_\alpha]}{d\alpha}|_{\alpha=0} = \frac{d\sigma(\vec{x}_\alpha(1),\vec{0})}{d\alpha}|_{\alpha=0} = \partial_{(p)b}\sigma(p,q)\frac{dx^b_\alpha(1)}{d\alpha}|_{\alpha=0}$. Noticing that $\frac{d\vec{x}_\alpha(1)}{d\alpha}|_{\alpha=0}$ is arbitrary, one has (33).
Finally, let us consider the bi-scalar called van Vleck-Morette determinant. In a real manifold either Riemannian or Lorentzian, the definition (5) can be rewritten, employing any coordinate system $\vec{z} = (z^1,\cdots,z^D)$ defined in an open totally normal (or geodesically convex) neighborhood $\mathcal{T}$ as

$$\Delta_{VVM}(x,y) := \frac{(-1)^D}{g(\vec{x})}\sqrt{\frac{g(\vec{x})}{g(\vec{y})}}\det\left(\frac{\partial^2\sigma(\vec{x},\vec{y})}{\partial x^a \partial y^b}\right)\,. \tag{35}$$

This expression can be generalized to open totally normal neighborhoods in complex pseudo-Riemannian manifolds. Notice that the *bi-scalar* so obtained is (complex) jointly-analytic in $\mathcal{T}\times\mathcal{T}$, but, in principle, can be a *multiple-valued* function due to the squared root. In any case, the branch point of the squared root is harmless since $g(\vec{z}) \neq 0$. Computing $\Delta_{VVM}(x,y)$ in normal coordinates centered in x (these coordinates do exist and cover $\mathcal{T}$ in our hypotheses), making use of (33), we get that, *in these coordinates*,

$$\Delta_{VVM}(x,y) = \sqrt{\frac{g(\vec{x})}{g(\vec{y})}}\,(\neq 0)\,. \tag{36}$$

Therefore, the bi-scalar $\Delta_{VVM}(x,y)$ cannot vanish anywhere and is positive either for a Riemannian or Lorentzian metric (all that not depending on the used coordinates!).

We are now able to state and prove the most important theorem for our goal (we omit the index λ in some notation for the sake of simplicity).

Theorem 2.4. *Let* $(\mathcal{M}, \mathbf{g})$ *be a* $(D = d+1)$*-dimensional Lorentzian manifold with class* C^ω*. Let* $\mathcal{O} \subset \mathcal{M}$ *be any open set endowed with (*C^ω*) local synchronous coordinates (with respect to some spacelike hypersurface)* $\vec{x} = (x^0, \cdots, x^d)$*. Fix a positive real* λ *and consider the set of complex pseudometrics* $\{\mathbf{g}_{\lambda\theta}\}$ *defined in* (13) of **Theorem 2.1** *in the analytically extended coordinates* $z^0, \cdots, z^d$ *(*$z^a = x^a + iy^a$*) varying in a open complex set* $\mathcal{G} \subset \mathbb{C}^D$ *with* $\mathcal{O} \subset \mathcal{G}$ and $\theta \in \mathbb{C}$.

(a) *For any* $p \in \mathcal{G}$*, there is a local base of the topology of* $\mathcal{G}$*,* $\{\mathcal{G}_{pj}\}_{j\in\mathbb{R}}$*, consisting of open totally normal, geodesically linearly convex neighborhoods of* p in common with *all of the complex pseudo-metrics* $\mathbf{g}_{(\lambda\theta)}$ *for* θ *which belongs to an open complex neighborhood of* $[0,\pi]$*,* $\mathcal{K}_p$*. Moreover, each* $\bar{\mathcal{G}}_{pj}$ *is also totally normal and geodesically linearly convex, with respect to all of the complex pseudometrics when* $\theta \in \mathcal{K}_p$*, and* $\bar{\mathcal{G}}_{pj} \subset \mathcal{G}_{pj'}$ *if* $j < j'$.

(b) *If* $p \in \mathcal{O}$*, posing (with obvious notations referred to the coordinates* $\vec{z}$*)* $\mathcal{U}_{pj} := Re\, \mathcal{G}_{pj}$*,* $\{\mathcal{U}_{pj}\}_{j\in\mathbb{R}}$*, is a local base of the topology of* $\mathcal{O}$ *about* p*, consisting of open totally normal, geodesically convex neighborhoods of the point* p *in common with whichever real (Riemannian or Lorentzian) metric produced, in the considered coordinates, by* $\mathbf{g}_{(\lambda\theta)}(\vec{x})$ *for particular choices of the, generally complex, value of* $\theta \in \mathcal{K}_p$*. In particular, this holds for the initial Lorentzian metric* $\mathbf{g}$ *(*$\theta = 0$*) and for the final Riemannian Wick-rotated metric* $\bar{\mathbf{g}}_\lambda$ *(*$\theta = \pi$*). Moreover, each* $\bar{\mathcal{U}}_{pj}$ *is also totally normal and geodesically convex, with respect all of the real metrics considered above and* $\bar{\mathcal{U}}_{pj} \subset \mathcal{U}_{pj'}$*, if* $j < j'$.

(c) *Arbitrarily fixed an element* $\mathcal{G}_{pj}$*, the* complex *functions obtained from* (30), (35) and (34) *specialized to the generic complex pseudometric metric* $\mathbf{g}_{(\lambda\theta)}$*,*

$$(\theta, q, q') \mapsto \sigma_\theta(q,q') \quad \textit{for} \quad (\theta,q,q') \in \mathcal{K}_p \times \mathcal{G}_{pj} \times \mathcal{G}_{pj} \tag{37}$$

$$(\theta, q, q') \mapsto \Delta^{1/2}_{VVM\theta}(q,q') \quad \textit{for} \quad (\theta,q,q') \in \mathcal{K}_p \times \mathcal{G}_{pj} \times \mathcal{G}_{pj} \tag{38}$$

$$(\theta, t, q, q') \mapsto \gamma_\theta(t,q,q') \quad \textit{for} \quad (\theta,t,q,q') \in \mathcal{K}_p \times B_2(0) \times \mathcal{G}_{pj} \times \mathcal{G}_{pj} \tag{39}$$

are jointly-analytic functions. Moreover, $\Delta^{1/2}_{VVM\theta}(\vec{x},\vec{y})$ *is a* single-valued *function and can be defined such that it coincides with the usual real positive van Vleck-Morette determinant for real metrics considered in* (b).

Proof. See the **Appendix**. □

3 The symmetry of Seeley-deWitt coefficients in smooth manifolds.

3.1. *The analytic case.* Let us consider the off-diagonal Seeley-deWitt coefficients given in **Definition 1.1**. It is possible to show that, if the Lorentzian metric and the function V are real analytic functions of the local coordinates, then the coefficients are symmetric functions of the arguments x and y. The way is direct, we can use the local Wick rotation previously defined and, via **Theorem 2.4**, we get the symmetry of the Seeley-deWitt coefficients from the symmetry of heat-kernel coefficients defined with respect the Wick-rotated Riemannian metric.

Theorem 3.1. *Let* $(\mathcal{M}, \mathbf{g})$ *be a (real, Hausdorff, paracompact, connected, orientable)* $(D = d+1)$*-Lorentzian* C^ω *manifold. Suppose the function* V *which appears in* (2) *is a (real) analytic function and consider the Seeley-deWitt/Hadamard coefficients given in* **Definition 1.1**.
Then, any point $p \in \mathcal{M}$ *admits a (totally normal) geodesically convex neighborhood* $\mathcal{N}_p$ *such that, if* $x, y \in \mathcal{N}_p$,

$$a_j(x,y) = a_j(y,x)\ , \tag{40}$$

for any $j \in I\!N$.

Proof. Fix any point $p \in \mathcal{M}$ and consider a synchronous coordinate system $x^{0,\cdots}, x^d$ defined in a open neighborhood $\mathcal{O}$ of p. Fix $\lambda = 1$ and use **Theorem 2.4** in $\mathcal{O}$ with respect to the coordinates $\vec{x}$. From now on, we shall use the notations of **Theorem 2.4**. Consider the local complex extension of the manifold defined on $\mathcal{G}$ and fix a common geodesically linearly convex set $\mathcal{H}_p = \mathcal{G}_{pj_0}$ of the local base of the topology found in (a) of **Theorem 2.4**. The Seeley-deWitt coefficients defined in $\mathcal{N}_p := Re\,\mathcal{H}_p$ by **Definition 1.1** can be analytically continued in the whole set $\mathcal{H}_p$. In particular we have that from **Definition 1.1** and (c) of **Theorem 2.4**, fixing the index j, and $x, y \in \mathcal{N}_p$, each function (with obvious notations)

$$\theta \mapsto a_j(\vec{x}, \vec{y}|\mathbf{g}_\theta) - a_j(\vec{y}, \vec{x}|\mathbf{g}_\theta)\ , \tag{41}$$

is analytic for $\theta \in \mathcal{K}_p$ where $\mathcal{K}_p$ is a complex open neighborhood of $[0, \pi]$. $\mathcal{K}_p$ can be assumed to be open and connected dropping the connected components which do not contain $[0, \pi]$. In particular, we can consider the complex values of θ, $\theta = \pi + i\mu$ where μ ranges in $[0, \epsilon[$. If ϵ is small enough, all these values of θ belong to $\mathcal{K}_p$. Then, we notice that, using the notations defined in (11) and (12), we have

$$\mathbf{g}_{(\lambda=1\,,\,\theta=\pi+i\mu)} = \mathbf{g}_{(\lambda=e^{-\mu/2}\,,\,\theta=\pi)}\ . \tag{42}$$

The metric on the right hand side is *Riemannian*.

Since the analytical continuation of the metric preserves the form of the right-hand side of (4), the analytical continuation of the off-diagonal Seeley-deWitt coefficients of the initial Lorentzian metric $\mathbf{g}$, for $\theta = \pi + i\mu$ produces the off-diagonal heat-kernel coefficients of the corresponding Riemannian metrics. Therefore, as we know by [Mo99c], the right hand side of (41) vanishes, whenever θ belongs to the set $\{\theta = \pi + i\mu \,|\, \mu \in [0, \epsilon[\} \subset \mathcal{K}_p$ for some $\epsilon > 0$. The uniqueness of the analytic continuation in open connected sets entails that the right hand side of (41) vanishes everywhere in $\mathcal{K}_p$, and in particular for $\theta = 0$. This means that the off-diagonal Seeley-deWitt coefficients defined with respect to the initial metric are symmetric functions of x and y in $\mathcal{N}_p$. $\square$

The result just proved implies the following more general result in a direct way, as remarked in **2.2** of [Mo99c].

Theorem 3.2. *Let $(M, \mathbf{g})$ be a (real, Hausdorff, paracompact, connected, orientable) $(D = d+1)$-Lorentzian C^∞ manifold. Consider the Seeley-deWitt Hadamard coefficients given in* **Definitions 1.1** *when both the metric* $\mathbf{g}$ *and the function V which appears in* (2) *are smooth fields.*
Then, any point $p \in \mathcal{M}$ admits a (totally normal) geodesically convex neighborhood $\mathcal{N}_p$ such that, if $x, y \in \mathcal{N}_p$,

$$a_j(x, y) = a_j(y, x)\,, \tag{43}$$

for any $j \in I\!N$.

Proof. The proof is exactly the same performed in the Riemannian case, **Theorem 2.2** in [Mo99c]. $\square$

Remark. This result can be achieved also if the manifold admits a smooth boundary as pointed out in [Mo99c].

Finally, we have a trivial corollary based on the fact that the Seeley-deWitt coefficients are also the coefficients which appear in the Hadamard local solution [Mo99c].

Corollary of Theorem 3.2. *Let $(M, \mathbf{g})$ be a (real, Hausdorff, paracompact, connected, orientable) $(D = d+1)$-Lorentzian C^∞ manifold. Let the metric* $\mathbf{g}$ *and the function V in* (2) *be smooth fields.*
Then, for any point $p \in \mathcal{M}$ there is a (totally normal) geodesically convex neighborhood $\mathcal{N}_p$ of p, such that, for any pair $(x, y) \in \mathcal{N}_p$, the coefficients u_j, v_j of the Hadamard parametrix, up to the order indicated (see (22) *and* (23) *in* [Mo99c]*),*

$$H_N(x, y) \;=\; \sum_{j=0}^{D/2-2} \left(\frac{2}{\sigma(x, y)}\right)^{D/2-j-1} u_j(x, y) + \sum_{j=0}^{N} \sigma^j(x, y) v_j(x, y) \ln(\sigma(x, y)/2) \tag{44}$$

($N \in I\!N$ fixed arbitrarily) for D even (the former summation appears for $D \geq 4$ only), and

$$
\begin{aligned}
H(x,y) &= \sum_{j=0}^{(D-5)/2} \left(\frac{2}{\sigma(x,y)}\right)^{D/2-j-1} u_j(x,y) + v_0(x,y)\sqrt{\frac{2\pi}{\sigma(x,y)}} \\
&\quad + v_1(x,y)\sqrt{2\pi\sigma(x,y)}
\end{aligned}
\tag{45}
$$

for D odd (the summation appears for $D \geq 5$ only), satisfy

$$
u_j(x,y) = u_j(y,x)\,, \tag{46}
$$

$$
v_j(x,y) = v_j(y,x)\,. \tag{47}
$$

3.2. *Final remarks.* The results proved in this work show that the Seeley-deWitt Hadamard off-diagonal coefficients are symmetric as requested within the point-splitting renormalization procedure of the stress-energy tensor. Such a result has been proved for the case where the manifold, the metric and the potential V are smooth. The result holds true in both Lorentzian and Riemannian manifolds. Anyway, the intriguing general fact we have pointed out is the existence of quite a natural local Wick rotation of the metric which preserves the local geodesical structures of the manifold making use of non-hermitian complex manifolds. This procedure makes sense regardless the presence of time-like Killing fields whenever the employed coordinates are somehow "physical". It is not so obvious what physics is involved in this procedure.

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Appendix: Proof of some Theorems and Lemmata.

Sketch of Proof of **Theorem 2.2**.
Fix a point $p \in \mathcal{S}$. Since $\mathcal{S}$ is embedded, it is possible to find a local coordinate system centered in p, $\vec{x} = (x^0, x^1 \cdots, x^d)$, defining a local chart $(\mathcal{U}_p, \vec{x})$ about p such that the set $\mathcal{U}_p \cap \mathcal{S}$ is given by the equation $x^0 = 0$. Then $(x^1, \cdots, x^d)$ define local *space-like* coordinates on $\mathcal{S}$ in a neighborhood of p. Now consider the local map

$(t, x^1, \cdots, x^d) \mapsto exp_{(0,x^1,\cdots,x^d)}(t\mathbf{N}(x^1, \cdots, x^d))$ which is defined in an open neighborhood of $(t = 0, x^1 = 0, \cdots, x^d = 0)$. $\mathbf{N}(x^1, \cdots, x^d)$ is the unique time-oriented vector normal to $\mathcal{S}$ in $(x^0 = 0, x^1, \cdots, x^d)$ with $\mathbf{g}(\mathbf{N}, \mathbf{N}) = -1$. It is a trivial task to compute the Jacobian determinant J_p of the map $(t, x^1, \cdots, x^d) \mapsto \vec{x}^{-1} \circ exp_{(0,x^1,\cdots,x^d)}(t\mathbf{N}(x^1, \cdots, x^d))$ at $(t = 0, x^1 = 0, \cdots, x^d = 0)$ obtaining $J_p = dx^0|_p(\mathbf{N}(p)) \neq 0$. Hence, the map $(t, x^1, \cdots, x^d) \mapsto exp_{(0,x^1,\cdots,x^d)}(t\mathbf{N}(x^1, \cdots, x^d))$ is a coordinate system in an open neighborhood of p, $\mathcal{U}'_p \subset \mathcal{U}_p$ and $(y^0, y^1, \cdots, y^d) := (t, x^1, \cdots, x^d)$ are local coordinates about $p \in \mathcal{S}$. Using the equation of geodesics, it is a trivial task to get that (9) and (10) are fulfilled and thus (making smaller $\mathcal{U}_p$, if necessary, in order to have a $\vec{y}$ domain of the form $]-\delta, \delta[\times\Omega$) we have built up a (time-oriented) locally synchronous coordinates with respect to $\mathcal{S}$. So local synchronous coordinates do exist. On the other hand it is also simply proved that the temporal coordinate of a point q in any (time-oriented) local synchronized coordinate system defined in **Definition 2.1** represents the (positive) length t_q of the *unique* geodesic segment which starts from $\mathcal{S}$ with a unitary initial tangent vector time-oriented normal to $\mathcal{S}$ at, say, $q' \in \mathcal{S}$ and reaches q. The spatial synchronous coordinates are nothing but the coordinates of q' on $\mathcal{S}$. Then, **Proposition 26** in Chap.7 of [ON83] entails that there is an open neighborhood $\mathcal{O}$ of $\mathcal{S}$ where any pair of geodesics starting from different points of $\mathcal{S}$ with initial tangent vector normal to $\mathcal{S}$ do not intersect each other anywhere (also if the starting points belong to *different* local synchronous coordinate system domains). By consequence, in $\mathcal{O}$, the temporal coordinate $q \mapsto t_q$ of any point q does *not* depend on the chosen local synchronous coordinate system. The coordinate transformation law between local synchronous coordinate system reads, in any common domain,

$$
\begin{aligned}
y'^0_q &= y^0_q = t_q \,, && (48)\\
y'^j_q &= y'^j_q(y^1_q, \cdots, y^d_q) \;,\; j = 1, \cdots, d \,. && (49)
\end{aligned}
$$

This trivially assures that the transformation law from different local synchronous coordinates preserves the form of the Wick rotated metric in common domains for any globally fixed value of λ. This defines a Riemannian metric on $\mathcal{N}$ which can be taken as the union of all af the intersections of $\mathcal{O}$ with each synchronous chart domain.

□

Proof of **Lemma 2.1**.
The differential equation system in **Lemma 2.1** is equivalent to the integral equation

$$Y(z, \bar{y}_0, \alpha) = \bar{y}_0 + \int_{z_0}^{z} f(u, Y(u, \bar{y}_0, \alpha), \alpha)du \tag{50}$$

where the path of integration is the segment from z_0 to z (Y is C^1 and thus analytic in z and the integration does not depend on the chosen path between the same extreme

points). We can write the equation above as

$$Y = A_{\bar{y}_0\alpha}(Y) \tag{51}$$

where $A_{\bar{y}_0\alpha}$ is defined by the right-hand side of (50) and it should be thought as a function which maps the Banach space $B := C^0(\bar{B}_{r_1}(z_0);\mathbb{C}^n)$ (with the norm $||\ \ ||_\infty$) into B itself. Actually $f(z, Y(z), \alpha)$ may not be defined, in general, when $Y \in B$ because some $Y(z)$ may be out of the domain of f. However, once one has fixed $r_2' > 0$ such that $r_2' < r_2$, taking a value $r_1' > 0$ which satisfies (17) one sees that $A_{\bar{y}_0\alpha}$ is well-defined on the the closed subset of B,

$$B_0 := \{Y \in C^0(\bar{B}_{r_1}(z_0);\mathbb{C}^n) \mid Y(z) \in \bar{B}_{r_2}(y_0) \textit{ for any } z\} \tag{52}$$

which is invariant under $A_{\bar{y}_0\alpha}$ provided $\bar{y}_0 \in \bar{B}_{r_2'}(y_0)$. In this domain, $A_{\bar{y}_0\alpha}$ is a contraction map, with contraction constant ρ, such that $0 < \rho < 1$ which does *not* depend on $\bar{y}_0 \in \bar{B}_{r_2'}(y_0)$ and $\alpha \in \bar{B}_{r_3}(\alpha_0)$, on that set. Banach theorem of the fixed point proves the existence and the uniqueness of the solution which is nothing but the fixed point of $A_{\bar{y}_0,\alpha}$ and belongs to B_0. In particular, the solution can be found as the limit (in the norm $||\,||_\infty$ with respect to the variable z, the remaining variables being fixed.)

$$Y = \lim_{k\to\infty} Y_k\,, \tag{53}$$

where $Y_k := A^k_{\bar{y}_0\alpha}(Y_0)$ and Y_0 is the constant function $z \mapsto Y_0(z) = y_0$ everywhere. Using the contraction property one finds that, for $k > m$,

$$Sup||Y_k(z,y,\alpha) - Y_m(z,y,\alpha)|| \leq \frac{\rho^{k-m}}{1-\rho}\, Sup||Y_1(z,y,\alpha) - Y_0(z,y,\alpha)||\,,$$

where the Sup is evaluated for $(z,y,\alpha) \in \bar{B}_{r_1'}(z_0) \times \bar{B}_{r_2'}(y_0) \times \bar{B}_{r_3}(\alpha_0)$. This entails that the convergence of the sequence (53) is uniform in all variables jointly. Since each function of the series is analytic by construction in any set $\bar{B}_{r_1'} \times \bar{B}_{r_2'}(y_0) \times \bar{B}_{r_3'}(\alpha_0)$, $0 < r_3' < r_3$, the limit function must be analytic therein. $\Box$

Proof of **Theorem 2.3**.
We follow and generalize the similar proof given in [KN63]. Let n the dimension of $\mathcal{M}$. Take a coordinate system centered in $p \in \mathcal{M}$, $\vec{z} = (z^1, \cdots, z^n)$, $p \equiv (0, \cdots, 0)$. We want to show that in these coordinates it is possible to find a class of open totally normal neighborhoods of p of the form $B_\rho(\vec{0}) := \{\vec{z} \in \mathbb{C}^n \mid \sum_{i=1}^n |z^i|^2 < \rho^2\}$, $0 < \rho < \bar{\rho}$, which are also linear geodesically convex and the class of the sets $\bar{B}_\rho(\vec{0})$ enjoys the same properties. The remaining part of the thesis is trivially proven by defining, for a fixed ρ, $0 < \rho < \bar{\rho}$, $\mathcal{G}_{pj} := B_{\rho(1+\tanh j)/2}(\vec{0})$ with $j \in \mathbb{R}$. The existence of the class above follows from a pair of propositions indicated by (p1) and (p2) in the following.

(p1) *Let* $S_\rho(\vec{0}) := \{\vec{z} \in \mathbb{C}^n \mid \sum_{i=1}^n |z^i|^2 = \rho\}$, $\rho > 0$, *then there exists* $c > 0$ *such that if* $\rho \in]0, c[$, *then any* real-parameter geodesic *which is tangent to* $S_\rho(\vec{0})$ *at a point, say* $\vec{y}$,

lies outside $S_\rho(\vec{0})$ in a neighborhood of $\vec{y}$.

Proof of (p1). Let $\vec{z} = \vec{z}(s)$, s defined in a neighborhood of s_0, be a real-parameter geodesic which is tangent to $S_\rho(\vec{0})$ at $\vec{y} = \vec{z}(s_0)$ (ρ will be restricted later). Formally speaking, this means that, setting

$$F(s) := ||\vec{z}(s)||^2 \,, \tag{54}$$

it holds $F(s_0) = \rho^2$ and

$$\frac{dF}{ds}|_{s=s_0} = \sum_i z^{*i}(s_0)\frac{dz^i}{ds}|_{s=s_0} + z^i(s_0)\frac{dz^{*i}}{ds}|_{s=s_0} = 0 \,. \tag{55}$$

Let us consider the second derivative of F at $s = s_0$. A trivial computation based on the derivation of central term in (55) and the equation of geodesics shows that

$$\frac{d^2F}{ds^2}|_{s=s_0} = V^\dagger A(\vec{z}, \vec{z}^*)V \tag{56}$$

where V is a vector with components $V^j = dz^j/ds|_{s=s_0}$ for $j = 1, \cdots n$ and * is the complex conjugation and † the hermitian conjugation. $A(\vec{z}, \vec{z}^*)$ is a $2n \times 2n$ Hermitian matrix with

$$A(\vec{z}, \vec{z}^*)_{ij} = \delta_{ij} \tag{57}$$

for $i, j = 1, \cdots, n$ and $i, j = n+1, \cdots 2n$, and

$$A(\vec{z}, \vec{z}^*)_{ij} = -\sum_{a=1}^{n} z^a \Gamma^a_{j\,k-n}(\vec{z}) \tag{58}$$

for $i = 1, \cdots, n$ and $j = n+1, \cdots, 2n$, and finally

$$A(\vec{z}, \vec{z}^*)_{ij} = -\sum_{a=1}^{n} z^{*a} \Gamma^{*a}_{j-n\,k}(\vec{z}) \tag{59}$$

for $i = n+1, \cdots, 2n$. and $j = 1, \cdots, n$. The matrix A becomes the identity matrix for $\vec{z} = \vec{z}^* = \vec{0}$ and thus is positive definite. There is a neighborhood of $\vec{0}$ which can be chosen in the form of $B_c(\vec{0})$, where, by continuity, $A(\vec{z}, \vec{z}^*)$ is defined positive. In this neighborhood $d^2F/ds^2|_{s=s_0} > 0$ and hence $F(s) > \rho^2$ when $s \neq s_0$ belongs to a real neighborhood of s_0, provided $\rho \in]0, c[$ [9]. This conclude the proof of (p1). $\Box$.

(p2) *Choose a real $c > 0$ as in* (p1). *Then there exists a real a with $0 < a < c$ such that:* (1) *Any two points of $B_a(\vec{0})$ $(\bar{B}_a(\vec{0}))$ can be joined by a complex geodesic segment which lies in $B_c(\vec{0})$;* (2) *Each point of $B_a(\vec{0})$ $(\bar{B}_a(\vec{0}))$ has a normal coordinate neighborhood*

[9] In general, this is not true in a *complex* neighborhood of 0 as one could trivially check in $\mathcal{M} = \mathbb{C}$.

containing $B_a(\vec{0})$ *(*$\bar{B}_a(\vec{0})$*) and thus is a* totally normal *neighborhood.*

Proof of (p2). Consider $\mathcal{M}$ as a submanifold of $T(\mathcal{M})$ in a natural way and work in coordinates $\vec{z}$ defined above, notice that $p \equiv \vec{0}$. Set

$$\phi : \mathbf{X} \mapsto (q, exp\mathbf{X}) \quad \text{for } \mathbf{X} \in T_q(\mathcal{M}) \,. \tag{60}$$

In general, ϕ is defined only in a neighborhood of $\mathcal{M}$ in $T(\mathcal{M})$. Since the differential of ϕ at $(\vec{0} \equiv p, \mathbf{0})$ is nonsingular, there exist a neighborhood V of $(\vec{0}, \mathbf{0})$ in $T(\mathcal{M})$ and a positive number $b < c$ such that ϕ defines a diffeomorphism in V onto $B_b(\vec{0}) \times B_b(\vec{0})$. Taking V and b small, we can assume that $exp(t\mathbf{X}) \subset B_c(\vec{0})$ for all $\mathbf{X} \in V$ and $t \in \mathbb{C}$, $|t| \leq 1$. The item (1) holds true for any $a > 0$ with $a \leq b$, since the complex geodesic segment from $q \in B_b(\vec{0})$ to $q' \in B_b(\vec{0})$ is the map $t \mapsto exp(t\mathbf{X})$ where $|t| \leq 0$ and $\mathbf{X} := \phi^{-1}(q, q') \in V$.
Let us consider the item (2). Fixed the positive real b and V as those in the proof of item (1), choosing $b' > 0$ and $\delta > 0$ small enough, we can fix an open subset of V, which is a neighborhood of $\vec{0}$ in $T(\mathcal{M})$, with the form $B_{b'}(\vec{0}) \times B_\delta$, where $0 < b' < b$ and B_δ is an open ball of radius $\delta > 0$ and center in $\mathbf{0} \in \mathbb{C}^n$. All the tangent spaces $T_q(\mathcal{M})$, $q \in B_{b'}(\vec{0})$, have been identified with $\mathbb{C}^n$ by means of the bases induced by the considered coordinates. Then, choose an open neighborhood $B_{b''}(\vec{0}) \times B_{b''}(\vec{0}) \subset \phi(B_{b'}(\vec{0}) \times B_\delta)$. Finally notice that, if $q \in B_{b''}(\vec{0})$, since ϕ is a diffeomorphism in $B_{b'}(\vec{0}) \times B_\delta$, we have

$$\{q\} \times B_{b''}(\vec{0}) \subset \phi(\{q\} \times B_\delta) \,,$$

and in particular, from the definition of ϕ,

$$B_{b''}(\vec{0}) \subset exp_q(B_\delta) \,. \tag{61}$$

This means that $B_a(\vec{0})$ is a totally normal neighborhood if $a \leq b''$ (and (1) also holds true due to $b'' < b$).
The proof for the closure of the considered neighborhoods is trivial and is obtained by taking a smaller and noticing that $\bar{B}_{a'}(\vec{0}) \subset B_a(\vec{0})$ if $a' < a$. $\Box$

To complete the proof of the theorem, let $0 < \rho < a (< c)$ and let q, q' be any pair of points in $B_\rho(p)$ ($\bar{B}_\rho(p)$). Let $\vec{z} = \vec{z}(s)$, $s \in [0, 1]$ the real-parameter segment geodesic from q to q' in $B_c(p)$ (see (p1)). We shall show that this real-parameter segment geodesic lies completely in $B_\rho(p)$ ($\bar{B}_\rho(p)$). Consider the function $s \mapsto F(s)$ defined in (54) along this geodesic segment. Assume that $F(s) \geq \rho^2$ ($F(s) > \rho^2$) for some s, that is, $\vec{z}(s)$ lies outside $B_\rho(p)$ ($\bar{B}_\rho(p)$) for some s. Let s_0, $s_0 \in]0, 1[$, be the value for which F attains the maximum, say, $\rho_0^2 \geq \rho^2$ ($\rho_0^2 > \rho$). Then

$$0 = \frac{dF}{ds}|_{s=s_0} \,. \tag{62}$$

This means that the real-parameter geodesic segment is tangent to the sphere $S_{\rho_0}(p)$ at the point $\vec{x}(s_0)$. By the choice of ρ the considered real-parameter geodesic segment lies inside the sphere $S_{\rho_0}(p)$, contradicting (p1). $\Box$

Proof of **Theorem 2.4**.
First of all, we notice that the item (c) is a direct consequence of items (a) and (b), **Lemma 2.1** and the discussion which follows that lemma. Barring the item (a), the only not completely trivial fact is the statement concerning the possibility of defining $\Delta^{1/2}_{VVM\theta}$ as a single-valued function which coincides with the usual one when evaluated on any $\mathcal{U}_{pj}$ for real (Riemannian or Lorentzian metrics). We shall prove this result in the end of the proof of this theorem.
Let us prove the validity of items (a) and (b). The latter is a straightforward consequence of the former taking into account that the the initial, the Wick-rotated and any other real metric obtained for the corresponding values of θ when restricting to real coordinates produce real exponential maps end geodesics. (Therefore, with respect to the considered coordinates, the exponential map transforms vectors with real components onto points with real coordinates. And the real-parameter geodesic segments connecting pairs of points in any $Re\,\mathcal{G}_{pj}$ ($\overline{Re\,\mathcal{G}_{pj}}$) are *real* geodesic segments completely contained in $Re\,\mathcal{G}_{pj}$, ($\overline{Re\,\mathcal{G}_{pj}}$).) Then we have to prove item (a) only. To this end, we use the same proof of **Theorem 2.3** with the necessary modifications.

Proof of (a). Fix any $\lambda > 0$. (From now on, for the sake of simplicity, we omit the index λ where not strictly necessary.) Take the complex coordinate system considered in the hypotheses. We are free to move the origin of the coordinate in $p \in \mathcal{G}$ by a complex translation. Let $\vec{u} = (u^1, \cdots, u^D)$ the new coordinate system. Therefore $p \equiv (0, \cdots, 0)$. We want to show that, in these coordinates, it is possible to find a class of open totally normal neighborhoods of p, corresponding to the metric $\mathbf{g}_{(\lambda\theta)}$, of the form

$$B_\rho(p) := \{\vec{u} \in \mathbb{C}^D \mid \sum_{i=1}^{D} |u^i|^2 < \rho^2\}\,, \tag{63}$$

$0 < \rho < \bar{\rho}$, which are also linear geodesically convex and the class of the sets $\bar{B}_\rho(\vec{0})$ enjoys the same properties. Moreover all these properties of a *fixed* neighborhood are preserved *varying* θ in a complex open neighborhood of $[0,\pi]$, $\mathcal{K}_p$. The remaining part of the thesis is trivially proven by defining, for a fixed ρ with $0 < \rho < \bar{\rho}$,

$$\mathcal{G}_{pj} := B_{(1+\tanh j)\rho/2}(\vec{p})\,, \tag{64}$$

with $j \in \mathbb{R}$. The thesis follows from a pair of propositions indicated by (p1) and (p2) in the following.

(p1) *Let* $S_\rho(\vec{0}) := \{\vec{u} \in \mathbb{C}^D \mid \sum_{i=1}^{D} |u^i|^2 = \rho\}$, $\rho > 0$, *then there exist* $c > 0$ *and a complex open neighborhood of* $[0,\pi]$, $\mathcal{K}'_p$, *such that if* $\rho \in]0, c[$, *then any* real-parameter

geodesic *defined with respect to any fixed metric* $\mathbf{g}_{(\lambda\theta)}$ *with* $\theta \in \mathcal{K}'_p$ *which is tangent to* $S_\rho(\vec{0})$ *at a point, say* $\vec{y}_\theta$*, lies outside* $S_\rho(\vec{0})$ *in a neighborhood of* $\vec{y}_\theta$*.*

Proof of (p1). Fix $\theta \in [0, \pi]$ arbitrarily, let $\vec{u} = \vec{u}_\theta(s)$, s being defined in a neighborhood of $s_{\theta 0}$ and with respect to the metric $\mathbf{g}_{(\lambda\theta)}$, be a real-parameter geodesic which is tangent to $S_{\rho_\theta}(\vec{0})$ at $\vec{y}_\theta = \vec{u}_\theta(0)$ (ρ_θ will be restricted later). Formally speaking, this means that, setting

$$F_\theta(s) := ||\vec{u}_\theta(s)||^2 \,, \tag{65}$$

it holds $F_\theta(0) = \rho_\theta^2$ and

$$\frac{dF_\theta}{ds}|_{s=s_{\theta 0}} = \sum_i u_\theta^{*i}(s_{\theta 0})\frac{du_\theta^i}{ds}|_{s=s_{\theta 0}} + u_\theta^i(s_{\theta 0})\frac{du_\theta^{*i}}{ds}|_{s=s_{\theta 0}} = 0\,. \tag{66}$$

Let us consider the second derivative of F_θ at $s = s_{\theta 0}$. A trivial computation based on the derivation of central term in (66) and the equation of geodesics shows that

$$\frac{d^2F_\theta}{ds^2}|_{s=s_{\theta 0}} = V_\theta^\dagger A_\theta(\vec{u}_\theta, \vec{u}_\theta^*)V_\theta \tag{67}$$

where V_θ is a vector with components $V_\theta^j = du_\theta^j/ds|_{s=s_{\theta 0}}$ for $j = 1, \cdots D$ and * is the complex conjugation and † the hermitian conjugation. $A(\vec{u}_\theta, \vec{u}_\theta^*)$ is a $2D \times 2D$ Hermitian matrix with

$$A_\theta(\vec{u}, \vec{u}^*)_{ij} = \delta_{ij} \tag{68}$$

for $i, j = 1, \cdots, D$ and $i, j = D + 1, \cdots 2D$, and

$$A_\theta(\vec{u}, \vec{u}^*)_{ij} = -\sum_{a=1}^{D} u^a \Gamma^a_{(\theta)j\;k-D}(\vec{u}) \tag{69}$$

for $i = 1, \cdots, D$ and $j = D + 1, \cdots, 2D$, and finally

$$A_\theta(\vec{u}, \vec{u}^*)_{ij} = -\sum_{a=1}^{D} u^{*a} \Gamma^{*a}_{(\theta)j-D\;k}(\vec{u}) \tag{70}$$

for $i = D + 1, \cdots, 2D$. and $j = 1, \cdots, D$. The matrix A_θ becomes the identity matrix for $\vec{u} = \vec{u}^* = \vec{0}$ and thus is positive definite. Now we let the coefficient θ in A_θ vary in a neighborhood of the initial value and rename the variable θ by η. Due to the joint continuity of the connection coefficients, there is an open neighborhood of $(\theta, \vec{0})$ which can be chosen in the form of $B_{\delta_\theta}(\theta) \times B_{c_\theta}(\vec{0})$, where $A_\eta(\vec{u}, \vec{u}^*)$ is positive definited. In this neighborhood $d^2F_\eta/ds^2|_{s=s_{\theta 0}} > 0$ and hence $F_\eta(s) > \rho_\eta^2$ when $s \neq s_{\theta 0}$ belongs to a real neighborhood of $s_{\theta 0}$. This procedure can be performed for any point $\theta \in [0, \pi]$ obtaining a covering of this

set made by complex open balls $B_{\delta_\theta}(\theta)$. Since $[0,\pi]$ is compact also as a complex set, we can extract a finite covering made of balls centered on the points θ_i, $i = 1, \cdots, N$, whose union is a complex open neighborhood of $[0,\pi]$, $\mathcal{K}'_p$. Let $c = Min\{c_{\theta_i} \mid i = 1, \cdots, N\}$. If $\vec{z} \in B_c(\vec{0})$ and $\theta \in \mathcal{K}'_p$, we have $d^2F_\theta/ds^2|_{s=s_{\theta 0}} > 0$ and hence $F_\theta(s) > \rho^2_\theta$ when $s \neq s_{\theta 0}$ belongs to a real neighborhood of $s_{\theta 0}$, provided $\rho_\theta \in]0, c[$. This conclude the proof of (p1). $\Box$

(p2) *Choose a real* $c > 0$ *as in* (p1). *Then there exist a real* a *with* $0 < a < c$ *and a complex open neighborhood of* $[0,\pi]$, $\mathcal{K}''_p$ *such that:* (1) *Fixing any* $\theta \in \mathcal{K}''_p$, *any two points of* $B_a(\vec{0})$ $(\bar{B}_a(\vec{0}))$ *can be joined by a complex geodesic segment of the metric* $\mathbf{g}_{(\lambda\theta)}$ *which lies in* $B_c(\vec{0})$; (2) *Fixing any* $\theta \in \mathcal{K}''_p$, *each point of* $B_a(\vec{0})$ $(\bar{B}_a(\vec{0}))$ *has a normal coordinate neighborhood, with respect to the metric* $\mathbf{g}_{(\lambda\theta)}$, *containing* $B_a(\vec{0})$ $(\bar{B}_a(\vec{0}))$, *and thus* $B_a(\vec{0})$ $(\bar{B}_a(\vec{0}))$ *is a* totally normal neighborhood *with respect to any metric* $\mathbf{g}_{(\lambda\theta)}$.

Proof of (p2). Consider $\mathcal{G}$ as a submanifold of $T(\mathcal{G})$ in a natural way and work in coordinates $\vec{u}$ defined above, notice that $p \equiv \vec{0}$. Set

$$\Phi : (\theta, \mathbf{X}) \mapsto (\theta, q, exp_{(\theta)}\mathbf{X}) \quad \text{for } \mathbf{X} \in T_q(\mathcal{M}), \theta \in \mathbb{C}\ . \tag{71}$$

In general, Φ is defined only in a neighborhood of $\mathcal{G}$ in $T(\mathcal{G})$. Since the differential of Φ at $(\theta, \vec{0}, \mathbf{0})$ for $\theta \in [0,\pi]$ is nonsingular, there exist an open neighborhood V_θ of $(\theta, \vec{0}, \mathbf{0})$ in $\mathbb{C} \times T(\mathcal{G})$, a complex open neighborhood $B_{r_\theta}(\theta)$ of θ and a positive number $b_\theta < c$ such that Φ defines a diffeomorphism in V_θ onto $B_{r_\theta}(\theta) \times B_{b_\theta}(\vec{0}) \times B_{b_\theta}(\vec{0})$. Taking V_θ, r_θ and b_θ small, we can assume that $exp_{(\eta)}(t\mathbf{X}) \subset B_c(\vec{0})$ for all $\mathbf{X} \in V_\theta$, $t \in \mathbb{C}$, $|t| \leq 1$ and $\eta \in B_{r_\theta}(\theta)$. Then extract a finite complex covering of $[0,\pi]$ made by balls $B_{r_{\theta_i}}(\theta)$, $i = 1, \cdots, M$. Let $\mathcal{K}''_{p1}$ be the union of the sets of this finite covering. The item (1) holds true for any $a > 0$ with $a \leq b := Min\{b_{\theta_i} \mid i = 1, \cdots, M\}$, since the complex geodesic segment corresponding to the metric $\mathbf{g}_{(\lambda\theta)}$ with $\theta \in \mathcal{K}''_{p1}$ from $q \in B_b(\vec{0})$ to $q' \in B_b(\vec{0})$ is the map $t \mapsto exp_{(\theta)}(t\mathbf{X})$ where $|t| \leq 0$ and $(\theta, \mathbf{X}) = \Phi^{-1}(\theta, q, q') \in B_{r_{\theta_k}} \times B_{b_{\theta_k}}(\vec{0})$ for some $k \in \{1, \cdots, M\}$.
Let us consider the item (2). Fixed any $\theta \in [0,\pi]$ and the positive reals r_θ, b_θ and V_θ exactly as those in the proof of item (1), choosing $b'_\theta > 0$ and $\delta_\theta > 0$ small enough, we can fix an open subset of $B_{r_\theta}(\theta) \times V_\theta$, which is a neighborhood of $(\theta, \vec{0}, \mathbf{0})$ with the form $B_{r'_\theta} \times B_{b'_\theta}(\vec{0}) \times B_{\delta_\theta}$, where $0 < r'_\theta < r_\theta$, $0 < b' < b$ and B_{δ_θ} is an open ball of radius $\delta_\theta > 0$ and center in $\mathbf{0} \in \mathbb{C}^n$. All the tangent spaces $T_q(\mathcal{G})$, $q \in B_{b'}(\vec{0})$, have been identified with $\mathbb{C}^n$ trough the bases induced by the considered coordinates. Then, choose an open neighborhood $B_{r''_\theta}(\theta) \times B_{b''_\theta}(\vec{0}) \times B_{b''}(\vec{0}) \subset \Phi(B_{r'_\theta}(\theta) \times B_{b'}(\vec{0}) \times B_\delta)$. Finally notice that, if $q \in B_{b''_\theta}(\vec{0})$ and $\eta \in B_{r''_\theta}(\theta)$, since Φ is a diffeomorphism in $B_{r'_\theta}(\theta) \times B_{b'_\theta}(\vec{0}) \times B_{\delta_\theta}$, we have

$$\{\eta\} \times \{q\} \times B_{b''_\theta}(\vec{0}) \subset \Phi(\{\eta\} \times \{q\} \times B_{\delta_\theta})\ ,$$

and in particular, from the definition of Φ,

$$B_{b''_\theta}(\vec{0}) \subset exp_{(\eta)q}(B_{\delta_\theta})\,, \tag{72}$$

for any $\eta \in B_{r''_\theta}$. All that can be performed for any fixed $\theta \in [0,\pi]$. Therefore, as we done before, we can extract a finite covering of $[0,\pi]$ made by balls $B_{r''_{\theta_k}}(\theta)$, $k = 1,\cdots,L$ with union $\mathcal{K}''_{p2}$. Then put $b'' := Min\{b''_{\theta_k} \mid k = 1,\cdots,L\}$. Then $B_a(\vec{0})$ is a totally normal neighborhood, if $a \leq b''$, with respect to all the metrics $\mathbf{g}_{(\lambda\theta)}$ whenever $\theta \in \mathcal{K}''_{p2}$. In $\mathcal{K}''_p := \mathcal{K}''_{p1} \cap \mathcal{K}''_{p2}$, (1) also holds true due to $b'' < b$.
The proof for the closure of the considered neighborhoods is trivial and is obtained by taking a smaller and noticing that $\bar{B}_{a'}(\vec{0}) \subset B_a(\vec{0})$ if $a' < a$. $\square$

To complete the proof of the item (a), let $0 < \rho < a (< c)$ and let q, q' be any pair of points in $B_\rho(p)$ ($\bar{B}_\rho(p)$), $p \equiv \vec{0}$. Let $\vec{u} = \vec{u}_\theta(s)$, $s \in [0,1]$ the real-parameter segment geodesic from q to q' in $B_c(p)$ (see (p1)) computed with respect to the metric $\mathbf{g}_{(\lambda\theta)}$ with θ arbitrarily fixed in the complex open neighborhood of $[0,\pi]$ given by $\mathcal{K}_p := \mathcal{K}'_p \cap \mathcal{K}''_p$. We shall show that this real-parameter segment geodesic lies completely in $B_\rho(p)$ ($\bar{B}_\rho(p)$). Consider the function $s \mapsto F_\theta(s)$ defined in (65) along this geodesic segment. Assume that $F_\theta(s) \geq \rho^2$ ($F_\theta(s) > \rho^2$) for some s_θ, that is, $\vec{u}_\theta(s_\theta)$ lies outside $B_\rho(p)$ ($\bar{B}_\rho(p)$) for some s_θ. Let $s_{\theta 0}$, $s_{\theta 0} \in]0,1[$, be the value for which F_θ attains the maximum, say, $\rho_\theta^2 \geq \rho^2$ ($\rho_\theta^2 > \rho$). Then

$$0 = \frac{dF_\theta}{ds}|_{s=s_{\theta 0}}\,. \tag{73}$$

This means that the real-parameter geodesic segment is tangent to the sphere $S_{\rho_\theta}(p)$ at the point $\vec{x}(s_{\theta 0})$. By the choice of ρ the considered real-parameter geodesic segment lies inside the sphere $S_{\rho_\theta}(p)$, contradicting (p1).

To end the proof, let us prove that in any set $\mathcal{G}_{pj}$ and for $\theta \in \mathcal{K}_p$, the van Vleck-Morette determinant can be defined as a single-valued function which coincides with the ordinary van Vleck-Morette determinant for whatever value of θ such that the metric is real, in particular $\theta = 0, \pi$.
By (35), we can assume that $\Delta^{1/2}_{VVM\theta}$ is single-valued, if the functions defined in our coordinates by

$$\begin{aligned}
(\theta, \vec{x}, \vec{y}) \mapsto F(\theta, \vec{x}, \vec{y}) &:= \frac{g_\theta(\vec{x})}{g_\theta(\vec{y})}\,,\\
(\theta, \vec{x}, \vec{y}) \mapsto G(\theta, \vec{x}, \vec{y}) &:= \frac{(-1)^D}{g_\theta(\vec{x})} det\left(\frac{\partial^2 \sigma_\theta(\vec{x}, \vec{y})}{\partial x^a \partial y^b}\right)\,,
\end{aligned}$$

take values away from the cut of a folder of the domain of definition of the functions $z \mapsto z^{1/4}$ and $z \mapsto z^{1/2}$. From now on, we fix this cut along the negative real axis and

work in the folder where both the functions $z \mapsto z^{1/4}$ and $z \mapsto z^{1/2}$ produce *real and positive* values when evaluated on *positive real* numbers. Let us prove that we can shrink the open set $B_\rho(p)$ in (63) (used to define the class of $\mathcal{G}_{pj} := B_{(1+\tanh j)\rho/2}(p)$) and $\mathcal{K}_p$ such that, in $\mathcal{K}_p \times B_\rho(p) \times B_\rho(p)$, the functions F and G take values with strictly positive real part.
Fix $\theta \in [0,\pi]$ and $p \equiv \vec{z}$ (the arbitrary center of $B_\rho(p)$). Trivially $F(\theta,\vec{z},\vec{z}) = 1$. On the other hand it also holds $\Delta_{VVM\theta}(\vec{z},\vec{z}) = 1$, because the VVM determinant is a bi-scalar and this result can be trivially obtained in normal coordinates centered in $\vec{x}$ by (36). Therefore we have also, in our coordinates, $G(\theta,\vec{z},\vec{z}) = 1$. Since F and G are jointly continuous in $(\theta,\vec{x},\vec{y})$, there is a neighborhood of $(\theta,\vec{z},\vec{z})$ of the form $B_{k_\theta}(\theta) \times B_{\rho_\theta}(p) \times B_{\rho_\theta}(p)$, $0 < \rho_\theta \leq \rho$, where both the functions assume only values with strictly positive real part. This procedure can be performed for any point $\theta \in [0,\pi]$, obtaining a covering of this set made of complex open disks $B_{k_\theta}(\theta)$. By compactness, we can extract a finite subcovering made by disks $B_{k_{\theta_i}}(\theta_i)$ centered in θ_i, $i = 1, \cdots, L$, and a corresponding finite class of open neighborhood of p, $B_{\rho_{\theta_i}}(p)$, $i = 1, \cdots, L$. Then we can take $\rho' > 0$ such that $B_{\rho'}(p) \subset \cap_i B_{\rho_{\theta_i}}(p)$ and use $B_{\rho'}(p)$ to define the class of $\mathcal{G}_{pj}$ by (64). Finally $\mathcal{K}_p$ can be re-defined as the intersection between the initial $\mathcal{K}_p$ and $\cup_i B_{k_{\theta_i}}(\theta_i)$. With these definition both functions F and G assume values with strictly positive imaginary part whenever $\theta \in \mathcal{K}_p$ and $\vec{x}, \vec{y}$ belong to any $\mathcal{G}_{pj}$. This implies that, in any $\mathcal{G}_{pj}$, $\Delta^{1/2}_{VVM\theta}$ can be defined as a single-valued function for any $\theta \in \mathcal{K}_p$. Moreover, with the choice above of the folder of definition of the functions $z \mapsto z^{1/4}$ and $z \mapsto z^{1/2}$, $\Delta^{1/2}_{VVM\theta}(\vec{x},\vec{y})$ coincides to the usual real one for these θ such that the function takes real values. $\Box$

Note added.

S. Hollands pointed out to me that a partial but relevant result about the symmetry of Lorentzian Hadamard coefficients $v_j(x,y)$ in (44) is contained in the final chapter of Friedlander's book, *The wave equation on a curved space-time* (Cambridge University Press, Cambridge, 1975). This result gives an overlap with results of the present work in a direct corollary of Theorem 6.4.1 in Fiedlander's book and taking into account the comment after Theorem 4.3.1 where it is specified that the coefficients considered in the book essentially coincide with $v_j(x,y)$ Hadamard's coefficients despite a different use and definition. This corollary and the comment show that, in a smooth Lorentzian manifold, when x, y belong to a common, sufficiently small, *causal domain* and $\sigma(x,y) < 0$ is satisfied, then $v_j(x,y) = v_j(y,x)$. This result is achieved using the whole theory of Lorentzian distribution developed in the book (see in particular Theorems 5.2.1 and 6.3.2) and makes use of the *method of descent* which explicitly requires a *Lorentzian* (D dimensional) metric. Finally, within a short remark after Theorem 6.4.1, it is suggested that a generalization to a whole causal domain may be obtained in the analytic case. Then, it is argued that the smooth non-analytic case also could be enconpassed by means os somehow approxi-

mation procedure of smooth differential equations by analytic differential equations. This last part of the suggested procedure seems to be exactly what we explicitly done in **Theorem 3.2**.
I am very grateful to S. Hollands for his remark.

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